A numerical study of the spectral gap

2008 J. Phys. A: Math. Theor. 41 055201

(http://iopscience.iop.org/1751-8121/41/5/055201)

J. Phys. A: Math. Theor. **41** (2008) 055201 (19pp) doi:10.1088/1751-8113/41/5/055201

A numerical study of the spectral gap

Pedro Antunes[1] **and Pedro Freitas**[2]

[1] CEMAT, Instituto Superior Técnico (TU Lisbon), Av.Rovisco Pais, 1049-001 Lisboa, Portugal
[2] Departamento de Matemática, Faculdade de Motricidade Humana (TU Lisbon) and Group of Mathematical Physics of the University of Lisbon, Complexo Interdisciplinar, Av. Prof. Gama Pinto 2, P-1649-003 Lisboa, Portugal

E-mail: pant@math.ist.utl.pt and freitas@cii.fc.ul.pt

Received 6 August 2007, in final form 13 December 2007
Published 23 January 2008
Online at stacks.iop.org/JPhysA/41/055201

Abstract

We present a numerical study of the spectral gap of the Dirichlet Laplacian, $\gamma(K) = \lambda_2(K) - \lambda_1(K)$, of a planar convex region K. Besides providing supporting numerical evidence for the long-standing gap conjecture that $\gamma(K) \geqslant 3\pi^2/\mathrm{d}^2(K)$, where $\mathrm{d}(K)$ denotes the diameter of K, our study suggests new types of bounds and several conjectures regarding the dependence of the gap not only on the diameter, but also on the perimeter and the area. One of these conjectures is a stronger version of the gap conjecture mentioned above. A similar study is carried out for the quotient of the first two Dirichlet eigenvalues.

PACS numbers: 02.30.Jr, 02.70.Hm
Mathematics Subject Classification: 35P15, 58G25

1. Introduction

This is the second in a series of papers consisting of a numerical study of several issues related to the spectrum of the Laplace operator. The main purpose of this program is to unveil some of the structure behind the connection between the low eigenvalues and certain elementary geometric quantities such as the perimeter, the area and the diameter. Many relations of this type are, of course, known, but it is our belief that they may still be improved in many cases and that at this stage this is best done with the help of numerical insight. This is due to the fact that some of the expressions obtained are quite involved, although they appear quite naturally when seen from the appropriate point of view—see, for instance, conjectures 7, 10, 15 and 17 below.

To illustrate our point, and also because of the connection with the present work, let us consider one of the conjectures that resulted from our previous work [AF]. There we considered the first Dirichlet eigenvalue and studied this quantity on polygons. This led us to the following conjecture which, if true, is an improvement of a result by Payne and Weinberger [PW].

Conjecture 1. *For any planar simply connected domain* Ω *we have*

$$\lambda_1(\Omega) \leqslant \frac{\pi j_{01}^2}{A} + \frac{\pi^2}{4}\frac{L^2 - 4\pi A}{A^2},$$

where L and A denote the perimeter and the area of Ω*, respectively.*

Inequalities of this type are not surprising in themselves, as they are simply a translation of the fact that there is an obvious relation between eigenvalues and the geometrical quantities involved. However, most of the classical results such as the Faber–Krahn inequality are what we might call *static*, in that they do not take into account how far we are from the optimal set in geometric terms. In the above case, the natural way of measuring this is by considering the isoperimetric defect $L^2 - 4\pi A$, and so we looked for the possibility of bounding the first eigenvalue by an expression of the form

$$\frac{\pi j_{01}^2}{A} + C\frac{L^2 - 4\pi A}{A^2}, \tag{1}$$

where the constant C was to be determined. We remark that the introduction of such terms has a counterpart in classical geometry where, for instance, there exist improvements of the classical geometric isoperimetric inequality with defect terms, such as Bonnesen's inequality—see [BZ] for several examples of this type. For other types of correction terms see, for instance, [FK, M].

Note that at first it is not clear that an inequality for λ_1 involving the term (1) should exist, nor that it should provide a lower or an upper bound. One of the surprising results in [AF] was precisely the fact that, within each class of n-polygons, the expression above (with the isoperimetric defect for a general domain replaced by that for n-polygons) does seem to provide both lower *and* upper bounds for the first Dirichlet eigenvalue if the constant C is chosen appropriately—see [AF] for more details. It turned out that while the upper bound converged to that in conjecture 1 as n went to infinity, the lower bound seemed to converge to that which is given by the Faber–Krahn inequality.

Another key ingredient leading to conjecture 1 was the identification of the extremal sets, that is, sets for which the inequality becomes an identity. While the ball is an obvious solution to this problem, the fact that we have an extra parameter suggested that there should exist other extremal sets whose nature is not so obvious *a priori*—see [AF] for the conjectured extremal sets under the restriction to families of n-polygons. In the case of upper bounds of the form (1) for general simply-connected domains the other extremal sets are what might be called asymptotical extremal sets, namely, a rectangle for which one side length is kept fixed while the other goes to infinity. Note that balls and infinite strips are the only planar domains with a smooth boundary having constant curvature.

An important issue here is that the infinite strips described above which are asymptotical extremal sets in conjecture 1 are also asymptotical extremal sets for the following gap conjecture,

Conjecture 2. *For any planar convex domain K we have*

$$\gamma(K) := \lambda_2(K) - \lambda_1(K) \geqslant \frac{3\pi^2}{\mathrm{d}^2},$$

where d *denotes the diameter of K.*

This conjecture has a long history which may be traced back to [Be] in the context of a free boson gas confined in a region in n-dimensional Euclidean space by a container with hard walls, and has received much attention during the intervening time. The conjecture is known to be true in the case of planar domains which are convex and symmetric with respect to two perpendicular axis [BK, D]. For other progress on this in the case of general convex domains see [SWYY, Sm, YZ], for instance, and the report [A] which contains an extensive bibliography on the subject.

All of the above suggested the performance in the case of the gap of a study similar to that which had been carried out in [AF], and, in particular, to check if it is possible to obtain a bound where besides infinite strips we also get equality for other domains. At a more general level, the purpose of the present paper is to understand the behaviour of certain functionals of the first two Dirichlet eigenvalues. More precisely, we shall consider the Dirichlet spectral gap and also mention briefly the spectral quotient defined for a domain Ω in R^2 by

$$\xi(\Omega) := \frac{\lambda_2(\Omega)}{\lambda_1(\Omega)}. \tag{2}$$

Apart from the fact that the spectral gap plays an important role in several areas of mathematical physics, the two functionals γ and ξ were chosen as the simplest examples where there are two effects pulling in opposite directions and where we expect at least some of the behaviour to be quite different—recall, for instance, that while the gap conjecture only makes sense for convex domains, in the case of ξ we know that it satisfies a static inequality of the Faber–Krahn type, namely, $\xi(\Omega) \leqslant \xi(B)$. This was conjectured by Payne, Pólya and Weinberger [PPW] in 1956 and was finally proved by Ashbaugh and Benguria in [AB], approximately 35 years later. In connection to this, we point out that the paper [Si] contains some isoperimetric inequalities similar to this for the spectral gap and quotient of triangles. There are also several other functionals of this type which have been considered in the literature. In [LY], for instance, the authors carried out a numerical study where besides ξ they also considered the quotient λ_3/λ_1.

A general outcome of our work is that one should expect the existence of similar results to those for the first Dirichlet eigenvalue for both γ and ξ. More precisely, we conjecture that there exist inequalities depending on geometrical quantities such as the perimeter and the area or the diameter and the area for which the extremal sets are the same as those mentioned above for the first Dirichlet eigenvalue, namely, balls and asymptotically on infinite strips. Furthermore, our results show that indeed there seems to be a very close relationship between the spectral gap and the diameter. This can be seen from the fact that within the class of n-polygons the optimal polygon is the same as the optimal polygon for geometric isodiametric inequalities—see section 3.1 below. Finally, we also detected that certain types of isosceles triangles play a role as extremal sets under certain conditions, in the sense that they seem to bound the possible values of the gap and the quotient from above. We also identified other sets in these conditions—see sections 3 and 4.

The method of study employed here is similar to that used in [AF]. More precisely, we begin by analysing some cases where the quantities involved are known explicitly, and rewrite the eigenvalues as a function of the geometric quantities mentioned above. Since for some of the problems considered balls and rectangles correspond to extremal domains, this allows us to obtain expressions for possible bounds directly. In other situations, such as when we have dependence on the diameter, rectangles turned out not to be extremal domains but the explicit expression obtained for them was still useful to understand the form the bound should take. These explicit expressions are then checked against a large sample of randomly generated planar domains consisting mainly of polygons with a small number of sides—our database has over 50.000 randomly generated convex polygons, of which nearly one eighth are (non-degenerate) octagons. Based on the numerical evidence we either discard the expression as a possible bound, or, in the case of positive results, proceed to test it against polygons with a larger number of sides and, in some cases such as the bounds for ξ, against non-convex polygons. In other cases we analysed the *clouds of points* generated for the gap and the quotient for a given class of polygons as functions of the perimeter or the diameter, and proceeded to identify the domains corresponding to the points on the boundary of these sets. The necessary computations to deal with such a large number of domains are quite heavy, and the eigenvalue calculation was done using the method of fundamental solutions which allows for such a treatment—see [AA] for details. For a different numerical approach see [TB].

The organization of the paper is as follows. Section 2 contains a study of some basic properties of the gap, such as (lack of) monotonicity, unboundedness, etc. Section 3 contains the numerical study of the gap, where we address its dependence on diameter, perimeter and area. In section 4 we mention some results from a similar study for the quotient of the first two eigenvalues. Finally, in section 5 we discuss the results obtained.

2. Some basic results for the gap

2.1. Unboundedness

We begin by recalling that $\gamma(K)$ is not bounded from above neither among convex planar sets of fixed diameter, nor among those of fixed area. This may already be found in [Sm], and it is a direct consequence of the fact that, for sufficiently small β, the first and second eigenvalues of the circular sector of angle opening β and radius r, $S_{\beta,r}$, are given by $j^2_{\frac{\pi}{\beta},1}\big/r^2$ and $j^2_{\frac{\pi}{\beta},2}\big/r^2$, respectively. On the other hand, the zeros $j_{\nu,i}\,(i = 1, 2)$ have the following asymptotic expansions as ν approaches infinity [EF]:

$$j_{\nu,i} = \nu - \frac{a_i}{2^{1/3}}\nu^{1/3} + O(\nu^{-1/3}), \qquad i = 1, 2.$$

Hence

$$\gamma(S_{\beta,r}) = 2^{2/3}\frac{a_1 - a_2}{r^2}\left(\frac{\pi}{\beta}\right)^{4/3} + O(\beta^{-2/3}), \qquad \text{as } \beta \to 0,$$

where $a_1 \approx -2.338\,11$ and $a_2 \approx -4.087\,95$ are the first and second negative zeros of the Airy function of the first kind, respectively.

Note that for β smaller than $\pi/3$ we have $\mathrm{d}(S_{\beta,r}) = r$ independently of β, and so this provides an example where the gap converges to infinity while keeping the diameter fixed. To see that it also yields that the gap is unbounded in the case of the fixed area problem, note that $A(S_{\beta,r}) = \beta r^2/2$ and we should thus take r equal to $(2A/\beta)^{1/2}$ in order to keep the area constant. This gives

$$\gamma(S_{\beta,(\frac{2A}{\beta})^{1/2}}) = \frac{a_1 - a_2}{2^{1/3}A}\frac{\pi^{4/3}}{\beta^{1/3}} + O(\beta^{1/3}), \qquad \text{as} \quad \beta \to 0.$$

Similar results may be obtained for other domains such as isosceles triangles and rhombi—see [F]. Note that this unbounded behaviour for fixed area in the thin limit case depends on smoothness assumptions of the domain, as it has been shown in [BF] that under some smoothness constraints the quantity $A(\Omega_\varepsilon)\gamma(\Omega_\varepsilon)$ will remain bounded as ε goes to zero, where Ω_ε denotes a domain that is being shrunk in one direction.

None of the above means, of course, that there are no local maxima for the gap, and this will indeed be the case as we will see below. On the other hand, it shows that the behaviour of γ is indeed quite different from that of ξ, at least in this respect.

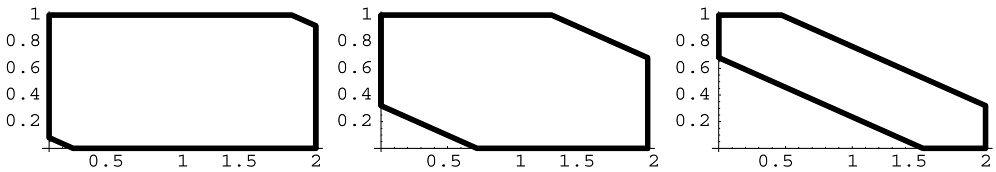

Figure 1. Three domains H_{t_i}, with $0 < t_1 < t_2 < t_3 < 1$.

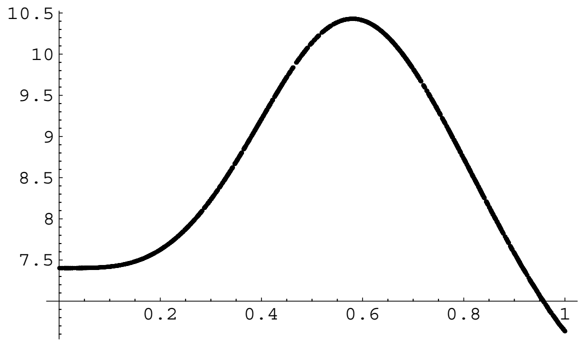

Figure 2. Plot of $\gamma\,(H_t)\,, t \in [0, 1]$.

2.2. *Dependence on the domain*

An obvious question that might be posed regarding the gap is whether or not it displays any sort of monotonic behaviour with respect to the domain. In general this is not to be expected, as the following simple example shows. Consider the unit square S and a rectangle R_ε which is inscribed in S with vertices a distance of $\sqrt{2}\varepsilon$ from two opposing vertices of the square for some positive ε. Then R_ε has sides of lengths 2ε and $\sqrt{2} - 2\varepsilon$ and we will assume that $2\varepsilon < \sqrt{2} - 2\varepsilon$. Then

$$\gamma(S) = 3\pi^2 \text{ and } \gamma(R_\varepsilon) = \frac{3\pi^2}{(\sqrt{2} - 2\varepsilon)^2},$$

from which it follows that although R_ε is always contained in S, $\gamma(S)$ will be larger or smaller than $\gamma(R_\varepsilon)$ depending on whether ε is larger than or smaller than $(\sqrt{2} - 1)/2$, respectively.

To illustrate a possible behaviour of the gap with respect to inclusion while keeping the diameter fixed, we shall consider a one-parameter family $H_t\,(t \in [0, 1])$ of domains of constant diameter for which $H_t \subset H_{t'}$ for $0 < t' < t < 1$ and such that the gap is increasing for t between zero and a value $T_1 \approx 0.58$, and then is decreasing for t up to 1. In figure 1 we plotted three domains H_{t_i}, with $0 < t_1 < t_2 < t_3 < 1$. In figure 2 we plotted $\gamma\,(H_t)\,, t \in [0, 1]$. These results show that, in general, the gap does not behave monotonically with respect to the domain.

3. Gap bounds

As mentioned in the introduction, we begin by considering the gap for classes of n-polygons. For simplicity in the statement of conjectures, we shall denote general domains by P_∞ and by P_n^{reg} and by P_∞^{reg} the regular n-polygon and the disk of unit area, respectively. We then have the following expression for the diameter of P_n^{reg}:

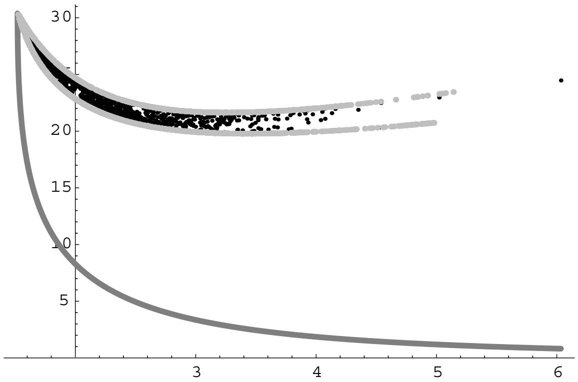

Figure 3. Plots of $\gamma(\mathrm{d})$ and the corresponding bound for triangles.

$$\delta_n := \mathrm{d}\left(P_n^{\mathrm{reg}}\right) = \begin{cases} \dfrac{2\sqrt{2}}{\sqrt{n \sin \frac{2\pi}{n}}} & \text{if} \quad n \text{ even} \\ \sqrt{\dfrac{2}{n \tan \frac{\pi}{2n} \cos \frac{\pi}{n}}} & \text{if} \quad n \text{ odd.} \end{cases} \tag{3}$$

Note that the sequence δ_n is not monotonic. On the other hand, computing the gap for regular n-polygons with unit area and n between 3 and 20 yields a decreasing sequence. Based on this we conjecture that the gap is decreasing among regular polygons with the same area.

Conjecture 3. *We have*

$$\gamma\left(P_3^{\mathrm{reg}}\right) > \gamma\left(P_4^{\mathrm{reg}}\right) > \cdots > \gamma\left(P_\infty^{\mathrm{reg}}\right).$$

3.1. Lower bounds depending on the diameter

Consider first bounds of the form

$$\frac{C}{\mathrm{d}^2}, \tag{4}$$

where C is a positive constant to be determined and which depends on the number of sides of the polygon. Since equality holds asymptotically for infinite strips, it is clear that conjecture 2 is optimal in the sense that if it holds it cannot be improved for a general polygon with a bound of the form (4). However, for triangles this is not necessarily the case and in fact our numerical study suggests the following:

Conjecture 4. *For any triangle T we have*

$$\gamma(T) \geqslant \frac{64\pi^2}{9\mathrm{d}^2}.$$

Equality holds if and only if T is an equilateral triangle.

In figure 3 we plotted γ as a function of the diameter and the corresponding bound for triangles with unit area. Isosceles triangles are marked in light grey and the bound with a continuous dark grey line. These results suggest that the gap has a local maximum at the equilateral triangle, and also that for fixed area its possible values for triangles are bounded from above and below by those of isosceles triangles. In what follows, we will need to distinguish between two types of isosceles triangles.

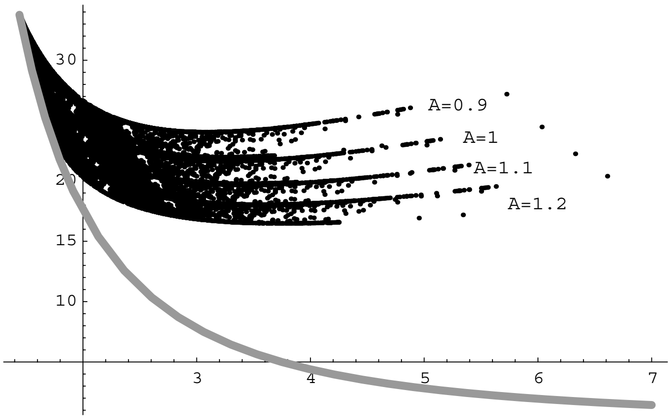

Figure 4. Plots of $\gamma(\mathrm{d})$ for triangles with different areas and the bound of conjecture 4.

Definition 3.1. *Let T be an isosceles triangle whose sides have lengths $l_1 \leqslant l_2 \leqslant l_3$. We will say that T is of type I if $l_1 \leqslant l_2 = l_3$ and that it is of type II if $l_1 = l_2 \leqslant l_3$.*

Conjecture 5. *For any triangle T we have*

$$\gamma(T_1) \leqslant \gamma(T) \leqslant \gamma(T_2),$$

where T_1 and T_2 are (respectively) isosceles triangles of type I and type II with the same area and diameter of T.

We shall now discuss the dependence of the bound in conjecture 4 on the area. Since for equilateral triangles we have $\gamma(T) = \lambda_2(T) - \lambda_1(T) = \frac{16\pi^2}{3\sqrt{3}A}$ and $A = \frac{\sqrt{3}\mathrm{d}^2}{4}$ it follows that equality holds in conjecture 4 for all equilateral triangles independently of the area. In figure 4 we plotted the bound of conjecture 4 and γ as a function of the diameter for triangles with areas $A = 0.9$, 1, 1.1 and 1.2. These results suggest that the bound of conjecture 4 cannot be improved within bounds of this type.

In line with the ideas mentioned in the introduction, we will now study bounds of a different type. For a rectangle R it is straightforward to obtain the expression

$$\gamma(R) = \frac{6\pi^2}{\mathrm{d}^2 + \sqrt{\mathrm{d}^4 - 4A^2}}, \tag{5}$$

where d and A denote the diameter and the area, respectively. This expression does not, however provide a lower bound for the gap. The reason for this is related to the isodiametric inequality for convex quadrilaterals, namely,

$$\mathrm{d}^2 \geqslant 2A. \tag{6}$$

While the square does provide equality in the above inequality, it is not the only quadrilateral to do so. In fact, there is a continuous family of quadrilaterals in the same situation and which will thus play a role in what happens at the far end where the diameter is minimal. It is thus relevant to study the gap for this family of domains. The numerical data gathered indicates that the square actually maximizes the gap, while the domain plotted in figure 5 minimizes it—all four lines marked with a dashed grey line have length equal to the diameter. In what follows we shall denote such quadrilaterals with area A by Q_{d}^A and in the case of unit area we shall simply write Q_{d}.

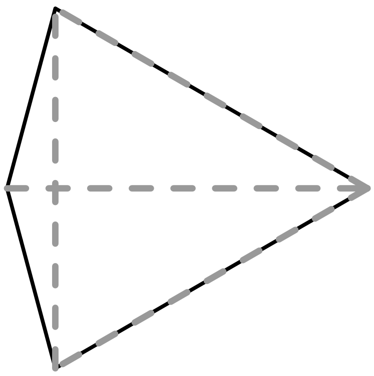

Figure 5. Quadrilateral with the same area and diameter of the square which minimizes the gap.

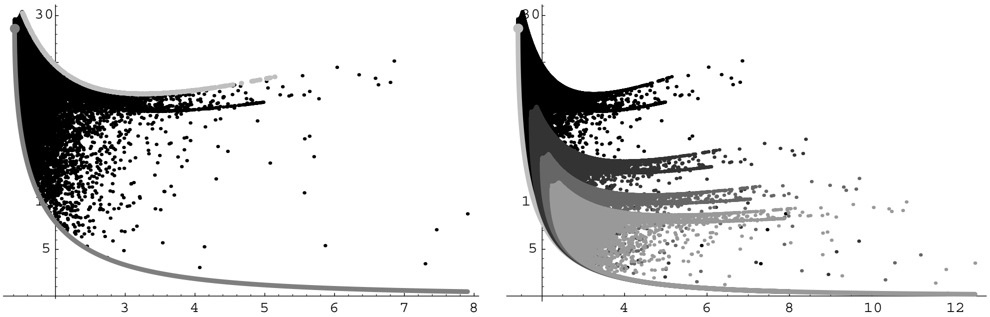

Figure 6. Plots of $\gamma(\mathrm{d})$ and the corresponding bound for quadrilaterals and the same plots for quadrilaterals with different areas.

The above discussion will now enable us to construct a lower bound for the gap. Inspired by the type of the expression obtained for rectangles we expect the following conjecture to hold.

Conjecture 6. *For any convex quadrilateral Q we have*

$$\gamma(Q) \geqslant \frac{6\pi^2\gamma(Q_{\mathrm{d}})}{3\pi^2\mathrm{d}^2 + [2\gamma(Q_{\mathrm{d}}) - 3\pi^2]\sqrt{\mathrm{d}^4 - 4A^2}}$$

with equality if and only if Q is a domain Q_{d}^A or asymptotically for infinite strips.

We plotted $\gamma(\mathrm{d})$ and the corresponding bound for quadrilaterals with unit area in the first plot of figure 6. The domain Q_{d} is marked with a larger grey point. We also plotted in light grey the gaps of isosceles triangles of type II (see conjecture 8 below). We shall now discuss the dependence of the inequality in conjecture 6 on the area. As we can see from figure 6, this effect is now not as dramatic as in the case of triangles. In particular, the gap of quadrilaterals of unit area already seems to become close to the conjectured bound. In the second plot of figure 6 we plotted $\gamma(\mathrm{d})$ for quadrilaterals with different areas ($A_1 = 1 > A_2 > A_3 > A_4$) in different shades of grey. We also plotted the bound in conjecture 6 obtained with area $A_i, i = 1, \ldots, 4$. This suggests that the bound obtained with $A = 1$ also provides a lower bound for a general convex quadrilateral with area $A > 1$. It is also clear that if we let the area go to zero, we then recover the bound in conjecture 2.

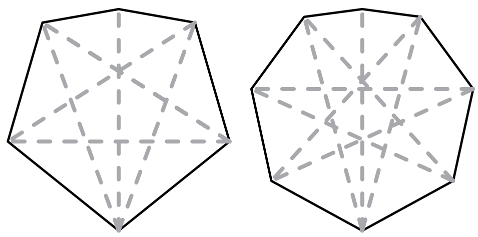

Figure 7. Polygons which minimize the diameter within the class of hexagons and octagons of fixed area.

Let us now consider the question of obtaining similar bounds within the class of convex n-polygons. From the analysis for quadrilaterals, it has become apparent that in order to do this it is necessary to know which polygons minimize the diameter within the class of n-polygons with unit area. This isodiametric problem has a long history dating back to [L] and is actually still open, but the results which are already known are sufficient for our purposes. The first important fact is that for odd n the regular polygon is the unique minimizer of the diameter within the class of convex n-polygons with fixed area [R]. The even case is not so straightforward, and the optimal domain is currently known only for values of n up to eight. In particular, it was also shown in [R] that for even n greater than or equal to six the regular polygon is never optimal. The case of hexagons was studied by Graham in [G], who obtained the hexagon which (for fixed area) minimizes the diameter. This optimal hexagon is shown in figure 7, where the lines with length equal to the diameter were marked with a dashed line. The optimal octagon was determined in [AHMX] (second plot of figure 7). In what follows we shall denote by $\mathcal{P}_n$ the optimal isodiametric n-polygon with unit area, and by β_n its diameter, that is, $\beta_n = \mathrm{d}(\mathcal{P}_n)$. We then propose the following conjecure:

Conjecture 7. *For any convex n-polygon P_n with $5 \leqslant n \leqslant \infty$ we have*

$$\gamma(P_n) \geqslant \frac{3\pi^2\beta_n^2\gamma(\mathcal{P}_n)}{3\pi^2\mathrm{d}^2 + \left[\beta_n^2\gamma(\mathcal{P}_n) - 3\pi^2\right]\sqrt{\mathrm{d}^4 - \beta_n^4 A^2}}$$

Equality holds only for optimal isodiametric polygons or asymptotically for infinite strips.

Remark 3.2. If we take n equal to 3 in conjecture 7 we obtain the bound

$$\gamma(T) \geqslant \frac{64\pi^2}{9\mathrm{d}^2 + \frac{37}{3}\sqrt{\mathrm{d}^4 - \frac{16A^2}{3}}}$$

and, as for triangles we have $\mathrm{d}^4 - \frac{16A^2}{3} \geqslant 0$, this bound is weaker than that of conjecture 4.

Remark 3.3. If we consider the degenerated case n equal to ∞ in conjecture 7, $\mathcal{P}_\infty$ is the ball of unit area and P_∞ shall denote any planar convex domain. In this case both a sharp isodiametric inequality and the optimal domain are known. More precisely, for such a domain P_∞ we always have [Bi]

$$\mathrm{d}^2(P_\infty) \geqslant \beta_\infty^2 A(P_\infty)$$

with $\beta_\infty = \frac{2}{\sqrt{\pi}}$ and equality if and only if P_∞ is a ball. Since $\beta_\infty^2\gamma(\mathcal{P}_\infty) - 3\pi^2 \approx 5.99 > 0$ it is clear that the bound proposed here is stronger than that in conjecture 2.

In figure 8 we plotted $\gamma(\mathrm{d})$ and the corresponding bound for convex n-polygons of unit area, with $n = 5, 6, 7, 8$. In each case we highlighted the n-polygon for which equality

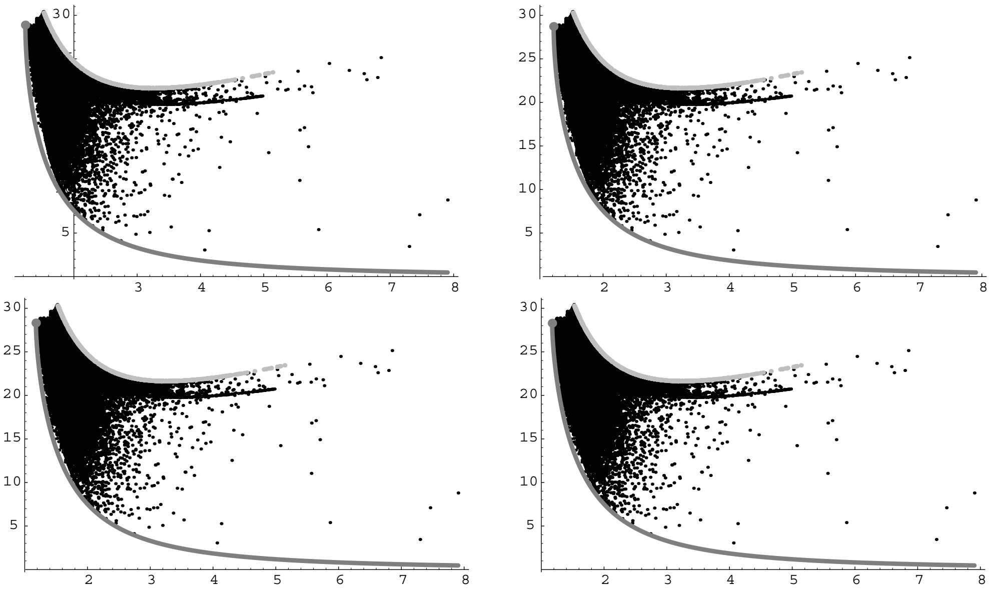

Figure 8. Plots of $\gamma(\mathrm{d})$ and the corresponding bounds for convex n-polygons of unit area, with $n = 5, 6, 7, 8$.

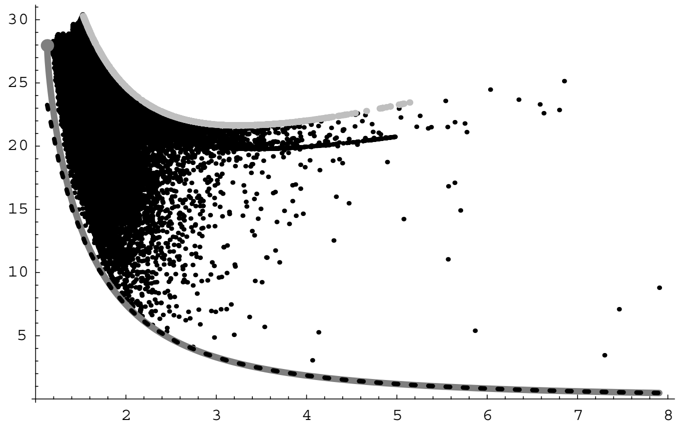

Figure 9. Plots of $\gamma(\mathrm{d})$, the bound of conjecture 7 for n equal to ∞ (with a continuous grey line) and that of conjecture 2 (with a dashed line).

holds with a grey point. We also marked in light grey the isosceles triangles of type II (see conjecture 8).

In figure 9 we plotted $\gamma(\mathrm{d})$, the bound of conjecture 7 for the case n equal to ∞ (with a continuous grey line) and that of conjecture 2 (with a dashed line) for convex domains with unit area.

3.2. Upper bounds depending on the diameter

We shall now consider the issue of obtaining upper bounds for the gap. The data for triangles which lead us to conjecture 4, suggest that the gap for a triangle T should be bounded above by that of isosceles triangles of type II with the same area and diameter of T. What we now claim is that this should be true also for a general convex domain K, provided that its diameter is larger than that of the equilateral triangle with the same area.

Conjecture 8. *Let K be a convex planar domain for which $\mathrm{d}^2(K) \geqslant \frac{4}{\sqrt{3}} A(K)$. Then*

$$\gamma(K) \leqslant \gamma(T_2),$$

where T_2 is the isosceles triangle of type II with the same area and diameter as K.

Remark 3.4. For domains for which the condition in the conjecture is not satisfied, there will exist no corresponding isosceles triangle with the same diameter and area. This means that the maximal gap under these circumstances has to be attained for a different type of domain. Although our data here does not allows us to be very definite, we conjecture that these domains have at least one axis of symmetry.

Remark 3.5. Clearly this bound depends on the diameter and the area only, except that in this case we do not have an explicit expression for the bound available, as we do not have a closed form for eigenvalues of isosceles triangles—for recent results along this direction, and in particular for the determination of the first terms in the asymptotic expansion of isosceles triangles of type II near the singular case, see [F].

3.3. Lower bounds depending on the perimeter

For a rectangle R it is quite straightforward to obtain the expression

$$\gamma(R) = \frac{48\pi^2}{(L + \sqrt{L^2 - 16A})^2}, \tag{7}$$

where L and A denote the perimeter and the area of R, respectively. The numerical results suggest that over all convex quadrilaterals with a given perimeter (and fixed area), the rectangle minimizes the gap. More precisely

Conjecture 9. *For any convex quadrilateral Q we have*

$$\gamma(Q) \geqslant \frac{48\pi^2}{(L + \sqrt{L^2 - 16A})^2}$$

with equality if and only if Q is a rectangle.

In order to state a general conjecture for n-polygons we need the isoperimetric constant for n-polygons, $\kappa_n = 4n\tan(\pi/n)$, that is, κ_n is such that for any n-polygon we have $L^2 \geqslant \kappa_n A$, with equality if and only if the polygon is regular. In the degenerated case n equal to ∞ the corresponding isoperimetric constant is $\kappa_\infty = 4\pi$.

Conjecture 10. *For any convex n-polygon P_n, with $3 \leqslant n \leqslant \infty$ we have*

$$\gamma(P_n) \geqslant \frac{\gamma\left(P_n^{\mathrm{reg}}\right)\kappa_n}{(L + \sqrt{L^2 - \kappa_n A})^2}.$$

Equality holds for regular n-polygons or asymptotically for infinite strips.

In figure 10 we plotted $\gamma(L)$ and the corresponding bounds for convex n-polygons of unit area, for $n = 3, 4, 5, 6, 7, 8$. Note that except for the case of quadrilaterals, there seems to be a gap between the cloud of points and the lower bound. This is because it is only in that case that we were able to identify the situation where identity holds for a given perimeter, and so

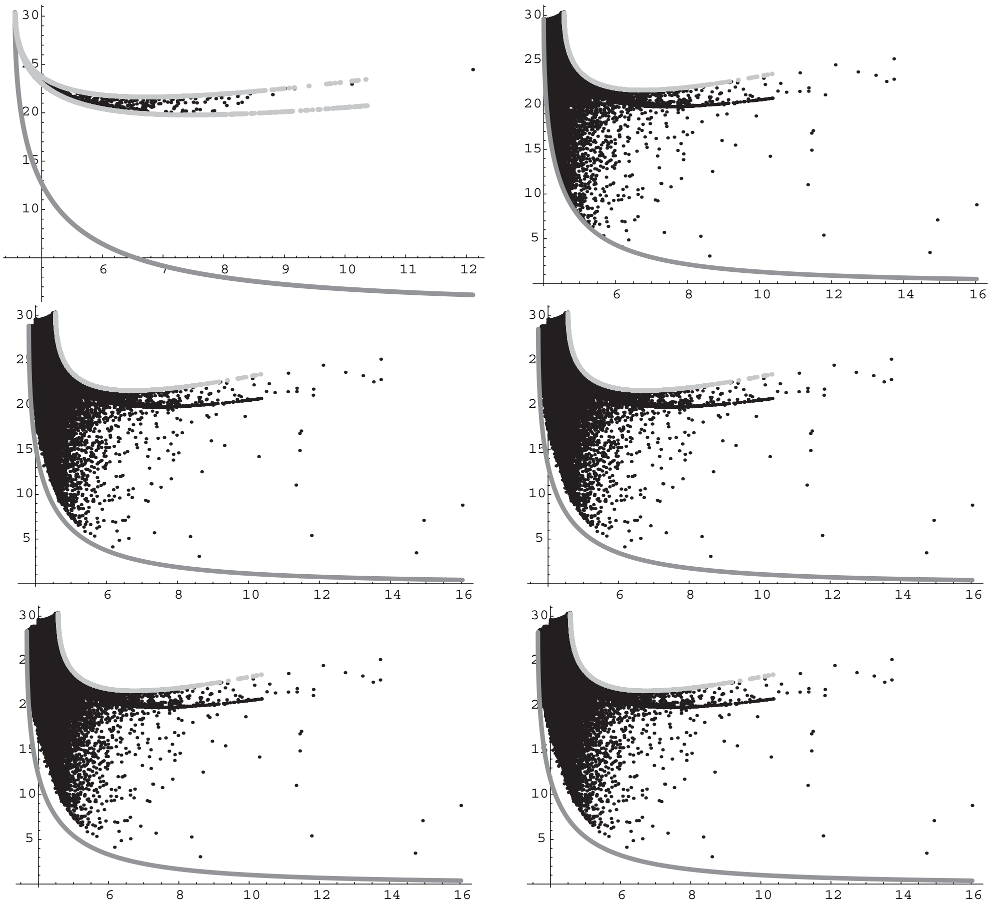

Figure 10. Plots of $\gamma(L)$ and the corresponding bounds for convex n-polygons of unit area, with $n = 3, 4, 5, 6, 7, 8$.

the bound is built from that example. Although infinite strips provide equality asymptotically, this will not be the case for rectangles when the perimeter is finite, if n is greater than four. In figure 11 we plotted $\gamma(L)$ for n-polygons ($3 \leqslant n \leqslant 8$) for convex domains with unit area together with the bound of conjecture 10 with n equal to ∞, and where we plotted isosceles triangles of type II in grey.

3.4. Upper bounds depending on the perimeter

As in the case of bounds depending on the diameter, we see that the gap is again bounded from above by that of isosceles triangles suggesting conjectures similar to conjectures 8 and 11. In the first plot of figure 10 we plotted $\gamma(L)$ for triangles with unit area (the gaps for isosceles triangles are plotted in light grey).

Conjecture 11. *For any triangle T we have*

$$\gamma(T_1) \leqslant \gamma(T) \leqslant \gamma(T_2),$$

where T_1 and T_2 are (respectively) isosceles triangles of type I and type II with the same area and perimeter as those of T.

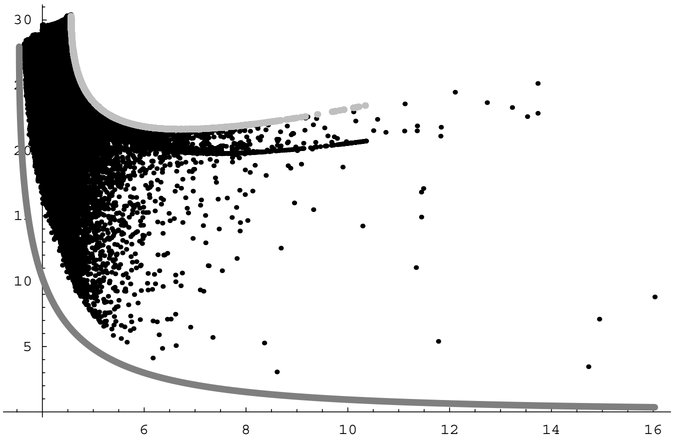

Figure 11. Plots of $\gamma(L)$ and the bound of conjecture 10 with n equal to ∞.

Conjecture 12. *Let K be a convex planar domain for which* $L^2(K) \geqslant 12\sqrt{3}A$. *Then*

$$\gamma(K) \leqslant \gamma(T_2),$$

where T_2 *is the isosceles triangle of type II with area A and perimeter L.*

4. Quotient bounds

In the case of the quotient ξ it is straightforward that the image of domains via ξ is the interval $(1, \xi(B))$. This follows from Ashbaugh and Benguria's theorem, and, on the other hand, from the fact that for rectangles where we make the length of one side go to infinity while keeping the other fixed we obtain that ξ converges to 1.

Regarding regular polygons, in this case we may go a bit further in that we might now expect the Ashbaugh–Benguria result to have a counterpart within the class of n-polygons, in the same way as the Pólya–Szegö conjectures relate to the Faber–Krahn inequality. In figure 12 we plotted $\xi(n)$ for n-polygons, and where each regular n-polygon is shown as a thicker dot. We propose

Conjecture 13. *The regular n-polygon maximizes* ξ *among all n-polygons.*

Note that in the same way as the Ashbaugh–Benguria result holds for general domains and not just in the convex case, here we also expect that to happen.

As in the case of the gap, the quotient is not monotonic with respect to inclusion. This is trivial to prove in this case, as it suffices to consider two balls B_1, B_2 and a square S such that we have $B_1 \subset S \subset B_2$. Since $\xi(B_1) = \xi(B_2) > \xi(S)$ the conclusion follows.

4.1. Lower bounds depending on the diameter

For a rectangle R we have the expression

$$\xi(R) = \frac{5}{2} - \frac{3}{2}\sqrt{1 - \frac{4A^2}{\mathrm{d}^4}}. \tag{8}$$

To obtain a lower bound we shall proceed in a similar fashion as in the case of the gap. We expect the following conjecure to hold:

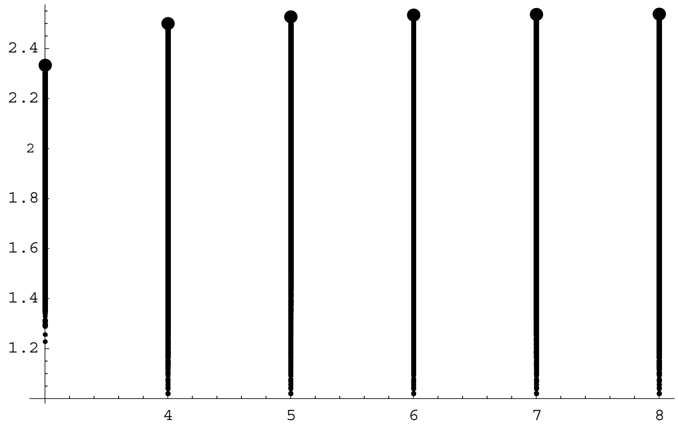

Figure 12. Plot of $\xi(n)$ for n-polygons.

Conjecture 14. *For any convex quadrilateral Q we have*

$$\xi(Q) \geqslant \xi\,(Q_{\mathrm{d}}) - \frac{3}{2}\sqrt{1 - \frac{4A^2}{\mathrm{d}^4}}$$

with equality if and only if Q is Q_{d}^{A}.

In the second plot of figure 13 we show $\xi(\mathrm{d})$ and the corresponding bound for convex quadrilaterals with unit area. The value obtained for $\xi\,(Q_{\mathrm{d}})$ is marked with a grey dot.

Conjecture 15. *For any convex n-polygon P_n with $5 \leqslant n \leqslant \infty$ we have*

$$\xi(P_n) \geqslant \xi\,(\mathcal{P}_n) - [\xi\,(\mathcal{P}_n) - 1]\sqrt{1 - \frac{\beta_n^4 A^2}{\mathrm{d}^4}}.$$

Equality holds for $\mathcal{P}_n$ or asymptotically for infinite strips.

In figure 13 we plotted $\xi(\mathrm{d})$ and the corresponding bound for convex n-polygons with unit area. In each case we also marked the polygon where equality holds. In figure 14 we plotted $\xi(\mathrm{d})$ for convex n-polygons with $3 \leqslant n \leqslant 8$ with unit area and the bound of conjecture 15 with n equal to ∞.

The numerical data that we gathered suggest that the values of the quocient ξ of triangles are bounded by those of the isosceles triangles as before.

4.2. *Lower bounds depending on the perimeter*

The expression for ξ in the case of rectangles may also be easily written in terms of the perimeter and the area as

$$\xi(R) = \frac{5}{2} - \frac{3L\sqrt{L^2 - 16A}}{2(L^2 - 8)}. \tag{9}$$

The numerical results suggest that over all the convex quadrilaterals with a given perimeter (and fixed area), the rectangle minimizes the quocient ξ. More precisely

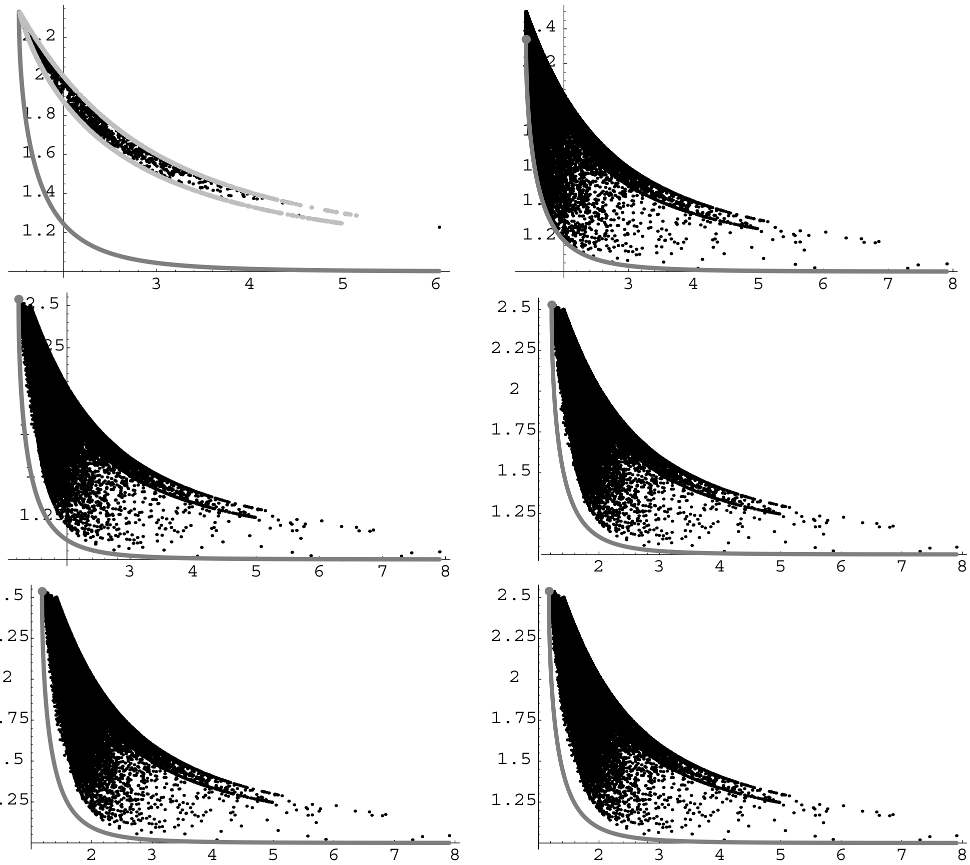

Figure 13. Plot of $\xi(\mathrm{d})$ and the respective bounds for convex n-polygons with $n = 3, 4, 5, 6, 7, 8$.

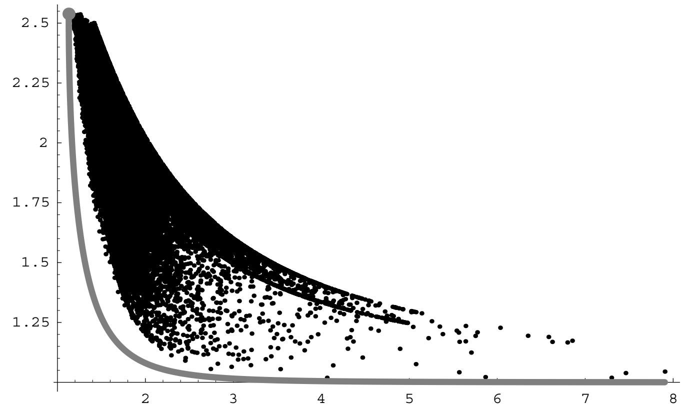

Figure 14. Plot of $\xi(\mathrm{d})$ for convex n-polygons with $3 \leqslant n \leqslant 8$ and the bound of conjecture 15 with n equal to ∞.

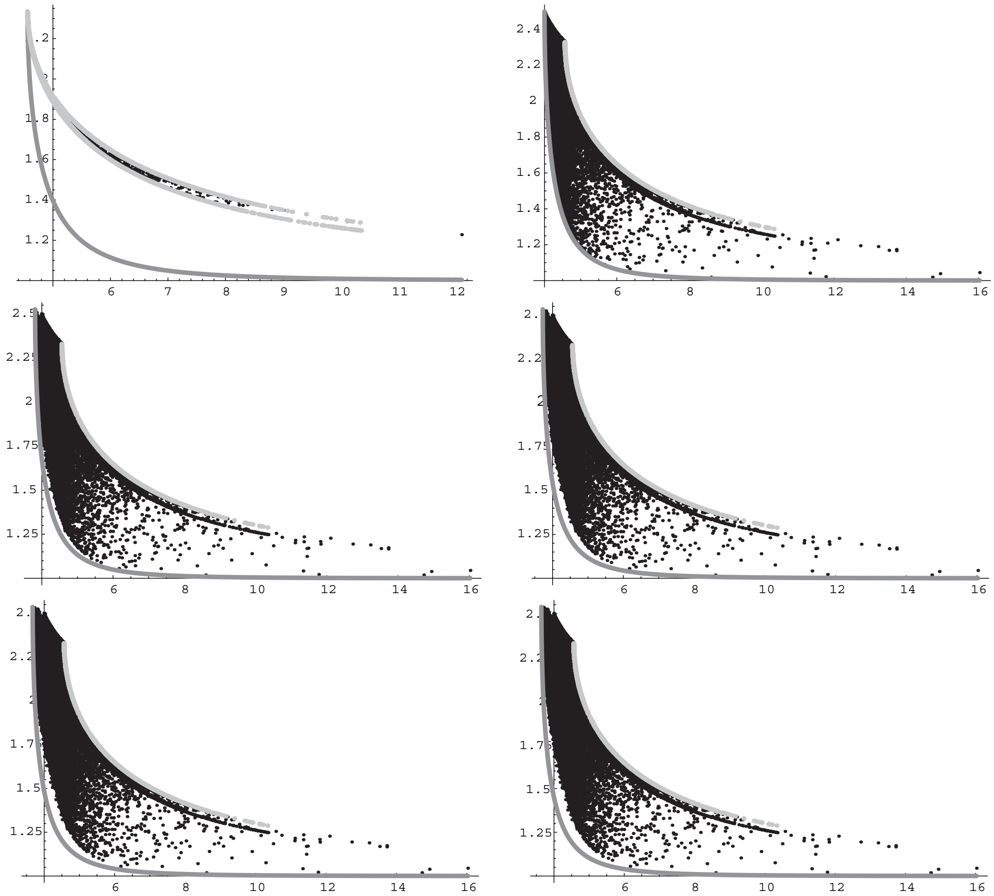

Figure 15. Plot of $\xi(L)$ and the respective bounds for n-polygons, with $n = 3, 4, 5, 6, 7, 8$.

Conjecture 16. *For any convex quadrilateral Q we have*

$$\xi(Q) \geqslant \frac{5}{2} - \frac{3L\sqrt{L^2 - 16A}}{2(L^2 - 8)}$$

with equality if and only if Q is a rectangle.

Proceeding as before now yields

Conjecture 17. *For a convex n-polygon P_n with $3 \leqslant n \leqslant \infty$ we have*

$$\xi(P_n) \geqslant \xi\left(P_n^{\mathrm{reg}}\right) - \frac{2\left[\xi\left(P_n^{\mathrm{reg}}\right) - 1\right]L\sqrt{L^2 - \kappa_n A}}{2L^2 - \kappa_n A}.$$

Equality holds for regular n-polygons or asymptotically for infinite strips.

In figure 15 we plotted $\xi(L)$ and the respective bounds for convex n-polygons with unit area, $n = 3, 4, 5, 6, 7, 8$.

In figure 16 we plotted $\xi(L)$ for convex polygons with $3 \leqslant n \leqslant 8$ and the bound of conjecture 17 with n equal to ∞.

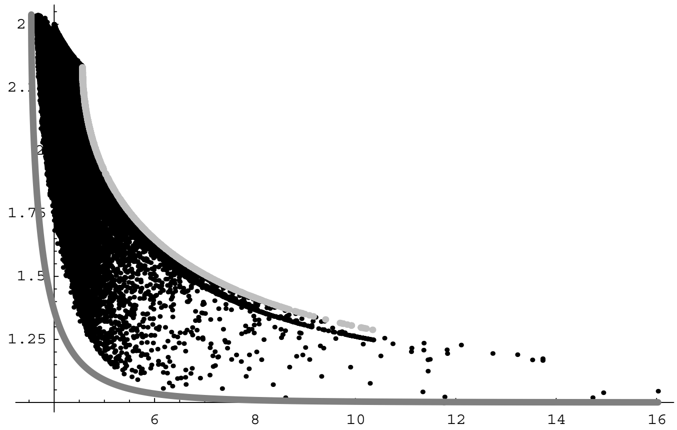

Figure 16. Plot of $\xi(L)$ and the bound of conjecture 17 with n equal to ∞.

5. Discussion

Our study enabled us to present and provide numerical support to a series of conjectures regarding the behaviour of two functions of the first two Dirichlet eigenvalues, namely, the spectral gap and quotient. By exploring explicitly the dependence on the area, these conjectures generalize the gap conjecture and propose extensions to existing results such as the Ashbaugh–Benguria theorem.

The numerical results obtained point to the existence of certain classes of extremal domains for which equality is attained in the isoperimetrical and isodiametrical inequalities conjectured. Although the appearance of some of these domains is not surprising, since infinite strips were already extremal sets for conjecture 2, by considering lower bounds which also take into account the area and not just the diameter, we point to a more general picture where balls and infinite strips are now both extremal sets for the same lower bound—see conjectures 7, 10, 15 and 17 for the case n equal to ∞. As pointed out in the introduction, this type of results is in line with those conjectured in [AF], where a similar phenomenon occurred. Furthermore, in this case and when considering upper bounds for the gap which depend both on the area and the diameter, for instance, there are other types of set which are extremal, such as isosceles triangles of type II , that is, those where the length of the equal sides is smaller than that of the third side, and rhombi. As far as we are aware, there were no previous results pointing in this direction.

Another important consequence was to highlight that the diameter of a convex domain does stand out as quite a natural quantity to include in lower bounds for the spectral gap, as the appearance of the optimal isodiametric polygons as extremal sets in conjecture 7 shows.

Finally, we would like to remark on the fact that in this study we looked mostly at convex domains. While in the case of the gap these are the natural domains to consider, when thinking about the spectral quotient one might expect the conjectures to extend to a wider class of domains as this is the case for the Ashbaugh–Benguria result. However, this turned out not to be the case, and it is possible to find nonconvex domains for which the spectral quotient is larger than that of the corresponding obtuse isosceles triangle with the same area and perimeter or area and diameter. As an example, one may take the right isosceles triangle T of unit area for which we have

$$\gamma(T) = \frac{5\pi^2}{2} \approx 24.674 \qquad \xi(T) = 2.$$

The quadrilateral Q with vertices at $(-a, 0)$, $(0, b)$, $(a, 0)$ and $(0, b+c)$ with $a = 1/c = 9/10$ and

$$b = -\frac{5}{9} + \frac{1}{5}\left(\frac{483\,033 + 144\,875\sqrt{2}}{640\,78}\right)^{(1/2)} \approx 0.0998,$$

has the same area and perimeter as the right isosceles triangle considered. However, $\gamma(Q) \approx 32.09$ and $\xi(Q) \approx 2.114$. If one takes instead the quadrilateral Q' with vertices at $(-\sqrt{2}, 0)$, $(0.1, 0)$, $(-0.05, 1/\sqrt{2})$ and $(0, \sqrt{2})$ we have that Q' has the same area and diameter as T but now $\gamma(Q) \approx 26.62$ and $\xi(Q) \approx 2.070$.

Although it is our belief that many of the conjectures presented here are probably beyond current available analytical techniques, we hope that this study will contribute to a better understanding of the behaviour of the two quantities involved, and, at a more general level, of the relations between eigenvalues and geometrical quantities such as those considered here.

Acknowledgments

Partially supported by FCT, Portugal, through programs POCTI/MAT/60863/2004 and POCTI/POCI2010.

References

[AA] Alves Carlos J S and Antunes Pedro R S 2005 The method of fundamental solutions applied to the calculation of eigenfrequencies and eigenmodes of 2D simply connected shapes, Computers *Comput. Mater. Continua* **2** 251–66

[AF] Antunes P and Freitas P 2006 New bounds for the principal Dirichlet eigenvalue of planar regions *Exp. Math.* **15** 333–42

[A] Ashbaugh M 2006 The fundamental gap, AIM 2006, available electronically at http://www.aimath.org/WWN/loweigenvalues/gap.pdf

[AB] Ashbaugh M and Benguria R 1992 A sharp bound for the ratio of the first two eigenvalues of the Dirichelt Laplacian and extensions *Ann. Math.* **135** 601–28

[AHMX] Audet C, Hansen P, Messine F and Xiong J 2002 The largest small octagon *J. Comb. Theory* A **98** 46–59

[BK] Bañuelos R and Kröger P 2001 Gradient estimates for the ground-state Schrödinger eigenfunction and applications *Comm. Math. Phys.* **224** 545–50

[Be] Berg M van den 1983 On condensation in the free-boson gas and the spectrum of the Laplacian *J. Stat. Phys.* **31** 623–37

[Bi] Bieberbach A 1915 Übber eine Extremaleigenschaft des Kreises *Jber. Deutsch. Math.-Vereinig* **24** 247–50

[BF] Borisov D and Freitas P 2007 Singular asymptotic expansions for Dirichlet eigenvalues and eigenfunctions on thin planar domains *Ann. Inst. H. Poincaré Anal Non Linéaire* at press

[BZ] Burago Yu D and Zalgaller V A 1988 *Geometric Inequalities* (Berlin: Springer)

[D] Davis B 2001 On the spectral gap for fixed membranes *Ark. Mat.* **39** 65–74

[EF] Elbert Á and Laforgia A 1998 Asymptotic expansion for zeros of Bessel functions and their derivatives for large order *Att. Sem. Mat. Fis. Univ. Modena* **XLVI** 685–95

[F] Freitas P 2007 Precise bounds and asymptotics for the first Dirichlet eigenvalue of rhombi and isosceles triangles *J. Funct. Anal.* doi:101016/jfa.2007.04.012

[FK] Freitas P and Krejčiřík D A sharp upper bound for the first Dirichlet eigenvalue and the growth of the isoperimetric constant of convex domains *Proc. Am. Math. Soc.* at press

[G] Graham R L 1975 The largest small hexagon *J. Comb. Theory* A **18** 165–70

[L] Lens H 1956 Ungelöste Probleme: Nr. 12 *Elem. Math.* **11** 86

[LY] Levitin M and Yagudin R 2003 Range of the first three eigenvalues of the planar Dirichlet Laplacian *LMS J. Comput. Math.* **6** 1–17

[M] Melas A 1992 The stability of some eigenvalue estimates *J. Differ. Geom.* **36** 19–33

[PPW] Payne L E, Pólya G and Weinberger H F 1956 On the ratio of consecutive eigenvalues *J. Math. Phys.* **35** 289–98

[PW] Payne L E and Weinberger H F 1961 Some isoperimetric inequalities for membrane frequencies and torsional rigidity *J. Math. Anal. Appl.* **2** 210–6

[R] Reinhardt K 1922 Extremale Polygone gegebene Durchmessers *Jber. Deutsch. Math.-Verein.* **31** 251–70

[SWYY] Singer I M, Wong B, Yau S-T and Yau S S-T 1985 An estimate of the gap of the first two eigenvalues in the Schrödinger operator *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **12** 319–33

[Si] Siudeja B 2007 Isoperimetric inequalities for eigenvalues of triangles *Preprint*

[Sm] Smits R 1996 Spectral gaps and rates to equilibrium in convex domains *Michigan Math. J.* **43** 141–57

[TB] Trefethen L N and Betcke T 2006 Computed eigenmodes of planar regions *Recent Advances in Differential Equations and Mathematical Physics (Contemp. Math.* vol 412*)* (Providence, RI: American Mathematical Society) pp 297–314

[YZ] Yu Q and Zhong J Q 1986 Lower bounds for the gap between the first and second eigenvalues of the Schrödinger operator *Trans. Am. Math. Soc.* **294** 341–9